Decay of Velocity and Temperature Fluctuations in Grid Turbulence

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The dependence of the velocity and temperature fluctuation decay exponents on the initial conditions is resolved on the basis of the cascade model for isotropic turbulence. In the large-Reynolds-number limit, self-preserving solutions are obtained for the nondissipative eddies. For those solutions, the energy decay exponent n and the small-wave-number exponent s for the three-dimensional energy spectrum obey the relation $n = 2(s + 1)/(s + 3)$. For temperature fluctuations, the decay exponent m and the small-wave-number spectrum exponent q satisfy $m = 2(q + 1)/(s + 3)$. Thus, the ratio $r = m/n$ of the thermal-to-mechanical time scales is completely determined by the initial spectral exponents for very large eddies, if the initial spectra are quasi-self-preserving. These conclusions are shown to be independent of the model. The model equations are solved numerically. The ratio of thermal to mechanical integral scales is shown to depend on both n and m. The conclusions have implications in the context of exact flow simulations.

I. Introduction

THE decay of isotropic turbulence, and of the fluctuations in scalar concentrations convected by it, can be the first step in the study of more complex flows. Of particular interest to the theoreticians, modelers, and experimentalists, power law decay of the variances seems to correspond to self-preserving spectral distributions.

Analytical studies of the decay laws and the experimental evidence are reviewed in Hinze.[1] For a suitable origin of dimensionless time, the evolution of velocity variance and integral length scale are described by the expressions

$$\overline{u^2}/u_0^2 = A(t)^{-n} \tag{1}$$

$$\mathcal{L}/\mathcal{L}_0 = B(t)^p \tag{2}$$

Because the dissipation rate can be estimated as $\epsilon \sim du^2/dt \sim u^3/\mathcal{L}$, the exponents n and p obey the relation

$$p = 1 - n/2 \tag{3}$$

Theoretical values of the exponents n and p are listed by Hinze (see Table 1) in the cases of exact self-preservation Batchelor-Townsend (B-T), Saffman invariance (S-I), and Loitsianskii invariance (L-I). The relation between the hypotheses leading to the L-I and S-I theoretical values and the observed decay of grid turbulence has not been established.

Experimental values are observed in the ranges

$$1 < n < 1.4, \qquad 0.3 < p < 0.53$$

with typical values of $n = 1.25$ and $p = 0.4$ in most experiments. Differences in the decay rates are attributed to the apparatus, without specific causal factors being proposed.

A number of experiments also document the decay of temperature fluctuations. Power-law curve fitting is again possible, under the form

$$\overline{\theta^2}/\theta_0^2 = C(t)^{-m} \tag{4}$$

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The ratio of the decay exponents for temperature and velocity variance

$$r = m/n \tag{5}$$

can be shown to measure the ratio of time scales for velocity and temperature.

Although it was conjectured that the parameter r could tend asymptotically to some universal value, Warhaft and Lumley[2] showed that r actually depends on the initial conditions (as represented by the relative length scales of injection of the temperature fluctuations) and that r remains approximately constant when power laws apply. Sreenivasan et al.[3] reported similar experiments, with r in a much narrower range of values. Even though it is obvious that the difference in the results can be attributed to the experimental conditions, no specific element of the apparatus has been associated with the observations in the cause-and-effect relationship.

Thus, the problem provides an opportunity for numerical studies. Newman and Herring[4] found that the test-field model yields a universal decay rate for the scalar variance, in contradiction with the experiments. Large-eddy simulation by Antonopoulos-Domis[5] obtained better agreeement, notably in showing a length-scale dependence for the scalar decay exponent m, but noted that m is not truly constant.

Newman et al.[6] showed good agreement with the data by modeling the evolution of r in terms of variances and dissipation rates. However, Pope[7] showed that the linearity of the scalar concentration equations and the assumption that the ratio r depends only on the variances and dissipation rates implies that the rate of change of r is incompatible with observations. Pope suggested that spectral considerations might resolve the issue.

Spectral resolution of the energy and scalar fields was adopted by a number of authors. The contributions of Nelkin and co-workers will be discussed further in the body of this paper. The EDQNM calculations of Vignon and Cambon[8] showed some discrepancies in the energy and temperature

Table 1 Theoretical power law exponents for several analytical constraints

	n	p
Batchlor-Townsend (B-T)	1	1/2
Saffman invariance (S-I)	6/5	2/5
Loitsianskii invariance (L-I)	10/7	2/7

spectra, as well as considerable errors in the predicted coefficients n and m, which the authors attribute to excessive spectral transfer in their model. Driscoll and Kennedy[9] extended Karman's model in their EVKS moel and obtained good agreement with Warhaft and Lumley's r and one-dimensional spectra. Durbin[10] used a stochastic mode to obtain good agreement with experiments, observing that m is not truly constant.

In this study, the cascade model of Desnyansky and Novikov[11] was used in spite of its known shortcomings[12] because of its simplicity. For grid turbulence, the model reduces to such a simple form that extensive exploration of the experimental parameters was convenient. Early numerical results pointed to the large-eddy spectral exponents as determining the decay exponents and provided the motivation for the analysis presented in section III. The analysis reveals that all conclusions based on the very-large-eddy spectral content are independent of the model.

II. Turbulent Energy Cascade

One of the recurrent notes in the preceding literature concerns the importance of length scales in the decay experiments. The cascade model (see Bell and Nelkin[13]) provides the simplest approach to a resolution of the turbulent energy spectrum. Although the model cannot be derived from the Navier-Stokes equations,[12] it includes some of the correct dynamics. Thus, it provides a simpler alternative to the numerical models discussed in Sec. I.

The wave numbers are discretized into octaves, with the factor 2 being the simplest nontrivial integer. The energy is then distributed among those modes; for simplicity, the cascade equations are written for the (scalar) velocities corresponding to the modal energies. The terms responsible for spectral transfer will be quadratic, by analogy with the Navier-Stokes equations. Additional requirements are that, with the exception of viscous dissipation, the energy should be conserved in the cascade, and that the energy is exchanged between eddies differing in scale only by a factor of 2. The resulting equation for the velocity μ at mode i is

$$\frac{d\mu_i}{dt} = \alpha k_i \left[(\mu_{i-1}^2 - 2\mu_{i+1}\mu_i) - 2^{1/3} C\,(\mu_{i-1}\mu_i - 2\mu_{i+1}^2) \right] - \nu k_i^2 \mu_i \tag{6}$$

where the terms in brackets account for spectral transfer, proportional to the dimensionless coefficient α; the constant C measures the relative importance of reverse spectral transfer (from small to large eddies); and ν is the kinematic viscosity governing the dissipation term. The proof of the conservation of energy by the transfer terms can be found in Ref. 11. Bell and Nelkin[13] studied the existence of a similarity region associated with the cascade. They showed that the inertial range exists for some values of the parameters α and C and that corrections to the Kolmogorov spectrum can also be calculated. They focused on the large- and small-k asymptotic cases. Bell and Nelkin[14] further studied the time dependence of the cascade spectrum under forcing and in decaying turbulence. They showed that power law decay is possible, with the exponent depending on the initial shape of the spectrum. They also proposed a dependence of the decay exponents on the reverse spectral transfer parameter C, which will be questioned later in the paper.

In connection with the cascade spectral velocity μ_i, a number of expressions can be listed for definition purposes and for eventual use.

Spectral energy density:

$$E_i(k_i) = v_i^2/k_i \tag{7}$$

Kolmogorov inertial constant:

$$E(k) = \alpha_k \epsilon^{2/3} k^{-5/3} \tag{8}$$

Root-mean-square velocity:

$$u' = \left(\frac{2}{3}\int E\,dk\right)^{1/2} \tag{9}$$

Dissipation rate:

$$\epsilon = 2\nu \int k^2 E\,dk \tag{10}$$

Integral scale:

$$\mathcal{L} = \frac{\pi}{2u'^2}\int \frac{E}{k}\,dk = \frac{3\pi}{4}\frac{\int \frac{E}{k}\,dk}{\int E\,dk} \tag{11}$$

Taylor microscale:

$$\lambda = \left(\frac{5\int E\,dk}{\int k^2 E\,dk}\right)^{1/2} = \left(\frac{15\nu u'^2}{\epsilon}\right) \tag{12}$$

Kolmogorov microscale:

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} = \left(\frac{\nu^2}{2\int k^2 E\,dk}\right)^{1/4} \tag{13}$$

Velocity microscale:

$$v = (\nu\epsilon)^{1/4} = (2\nu^2 \int k^2 E\,dk)^{1/4} \tag{14}$$

In the preceding equations, all integrals are from 0 to ∞. Because of the octave discretization of the wave-number axis, we have further, for example,

$$\int_0^\infty E(k)\,dk = \int_0^\infty \mu_i^2/k_i\,dk_i = \sum \mu_i^2\,d\log(k_i) = \log(2)\sum \mu_i^2 \tag{15}$$

III. Quasi-Self-Preserving Decay of Homogeneous Turbulence

A preliminary numerical study based on the cascade model indicated that a range of decay exponents can be obtained. Using ad hoc initial conditions for the spectrum, it was observed that power law decay corresponds to spectra that are self-preserving to a very good approximation. The model results also indicated that the slope of the spectrum in the small-wave-number limit determines the decay exponent. A similar result can be found in Leith.[15] Although these observations could have been model dependent, they provided the motivation for the following analysis.

Two scaling quantities are introduced for the purpose of removing the explicit time dependence from the equations: a velocity scale $U(t)$ and a length scale $L(t)$. Specific definitions of these scaling quantities will be introduced following Eq. (19), when their dynamic significance is more obvious. We have

$$\kappa_i = k_i \cdot L(t) \tag{16}$$

$$u_i(t, k_i) = v_i(\kappa_i) \cdot U(t) \tag{17}$$

Then the cascade equations become

$$\frac{du_i}{dt} = v_i \frac{dU}{dt} + U \frac{dv_i}{d\kappa_i}\frac{\kappa_i}{L}\frac{dL}{dt} = \alpha \frac{U^2}{L}\kappa_i(v_{i-1}^2 \ldots) - \nu \frac{u}{L^2}\kappa_i^2 v_i \tag{18}$$

where the term in parentheses is a shorthand for the inertial cascade terms. Multiplication by L/U^2 yields

$$\left(\frac{L}{U^2}\frac{dU}{dt}\right) v_i + \left(\frac{1}{U}\frac{dL}{dt}\right) \kappa_i \frac{dv_i}{d\kappa_i} = \alpha\kappa_i(v_{i-1}^2 \ldots) - \frac{\nu}{UL}\kappa_i^2 v_i \tag{19}$$

Self-preservation of the energy spectrum is expressed by a time-independent solution to Eq. (19), which is possible only if all coefficients of spectral quantities are proportional to each other. The well-known consequences of such assumptions are the $L \sim (\nu t)^{1/2}$, $E \sim t^{-1}$, and $Re = UL/\nu \sim t^0$. Although these solutions are confirmed by some data, the different exponents observed in most wind-tunnel experiments indicate that the self-preservation is stated in too restrictive terms.

Symptomatic of the constraint, the constant Reynolds number locks all turbulence length scales in a fixed ratio to one another. With a varying Reynolds number, quasi-self-preservation must result in at least two independent length scales. Thus, as noted in Hinze,[1] a partial self-preservation may be more representative of the observed decaying turbulence. De Kármán and Howarth[16] still obtained $n = 1$ by assuming self-preservation of the nondissipative eddies. However, a similar analysis by Leith[15] gave a range of possible decay exponents.

When $L \sim \mathcal{L}$ is taken to be the integral scale of the turbulence as defined by Eq. (11) and $U \sim u'$, the coefficient of the viscous dissipation term in Eq. (19) is the inverse of the large-eddy Reynolds number and is assumed to be asymptotically negligible: Quasi-self-preservation results from ignoring the time dependence of the very small viscous term. Then the equation becomes

$$\left(\frac{\mathcal{L}}{u'^2}\frac{du'}{dt}\right) v_i + \left(\frac{1}{u'}\frac{d\mathcal{L}}{dt}\right) \kappa_i \frac{dv_i}{\kappa_i} = \alpha\kappa_i(v_{i-1}^2 \ldots) - \frac{1}{Re}\kappa_i^2 v_i \tag{20}$$

Power law solutions of the form

$$\mathcal{L} = \mathcal{L}_0 \left(\frac{u_0}{\mathcal{L}_0} t\right)^p \tag{21}$$

$$u' = \mu_0 \left(\frac{u_0}{\mathcal{L}_0} t\right)^{-n/2} \tag{22}$$

can be substituted in Eq. (20), yielding

$$-\frac{n}{2}\left(\frac{u_0}{\mathcal{L}_0} t\right)^{p-1+n/2} v_i + \left(\frac{u_0}{\mathcal{L}_0} t\right)^{p-1+n/2} p\kappa_i \frac{dv_i}{d\kappa_i} = \alpha\kappa_i \times (v_{i-1}^2 \ldots) - \frac{1}{Re}\kappa_i^2 v_i \tag{23}$$

Quasi-self-preservation is possible with the assumed power laws (21) and (22), provided

$$p = 1 - n/2 \tag{24}$$

It follows that the large-eddy Reynolds number varies according to the power law

$$Re = \frac{u\mathcal{L}}{\nu} = Re_0 t^{p-n/2} = Re_0 t^{1-n} \tag{25}$$

For the Taylor microscale,

$$\lambda \sim \mathcal{L} Re^{-1/2} \sim t^{1/2} \tag{26}$$

irrespective of n.

For a fixed large Reynolds number, the self-preserving solutions obey the equation

$$\frac{1}{v_i}\frac{dv_i}{d\kappa_i} = \frac{n}{2-n}\frac{1}{\kappa_i} + \frac{2\alpha}{2-n}\frac{\kappa_i}{v_i}(v_{i-1}^2 \ldots) - \frac{2}{(2-n)Re}k_i^2 v_i \tag{27}$$

Clearly, a family of solutions exist, parameterized with the decay exponent n. For small or moderate wave numbers, the viscous term can be neglected. The κ_i^{-1} dependence of the second term suggests that it may dominate the large eddies. Assuming a power law dependence $v_i(\kappa_i) \sim \kappa_i^x$ as $\kappa_i \to 0$, the spectral transfer term is indeed negligible as long as $x > -1$. Under this condition, integration of Eq. (27) yields

$$v_i \to \kappa_i^{n/(2-n)} \quad \text{as} \quad \kappa_i \to 0 \tag{28}$$

or, for the three-dimensional energy spectrum,

$$e_i = v_i^2/\kappa_i \to \kappa_i^{(3n-2)/(2-n)} \tag{29}$$

Thus, the quasi-self-preserving spectrum decay exponent is directly related to the spectral exponent of κ_i as $\kappa_i \to 0$.

The question arises immediately of how much these results are model dependent. As underlined by the use of shorthand for the transfer terms, their analytical form does not in any way affect the derivations. All scaling and the subsequent reductions can be carried out on dimensional grounds. Thus, the quasi-self-similarity, the corresponding power law decay, and the small-wave-number spectrum [Eq. (29)] hold in general:

$$E_i(k_i, t) = u'^2\mathcal{L}e_i \sim k_i^{(3n-2)/(2-n)} \sim k_i^s, \qquad k_i \to 0 \tag{30}$$

Hinze[1] gives the corresponding exponents s for the B-T self-preserving spectrum and in terms of the L-I and Saffman (S-I) invariants (see Table 1). We must have, respectively,

B-T:

$$(3n-2)/(2-n) = 1 \quad \text{so} \quad n = 1 \tag{31}$$

L-I:

$$(3n-2)/(2n-n) = 4 \quad \text{so} \quad n = 10/7 \tag{32}$$

S-I:

$$(3n-2)/(2-n) = 2 \quad \text{so} \quad n = 6/5 \tag{33}$$

Thus, the relation between the decay exponent, the length scale exponent, and the small-k asymptotic energy spectrum is verified for those particular cases. Futhermore, the observed $n > 1$ corresponds to the existence of a first moment of the autocorrelation function, as seen from the integrability conditions

$$\int E\, dk < \infty \to s > -1 \quad \text{and} \quad 0 < n < 2 \tag{34}$$

$$\int \frac{E}{k} dk = \int B\, dr < \infty \to s > 0 \quad \text{and} \quad 2/3 < n < 2 \tag{35}$$

$$\int \frac{E}{k^2} dk = \int rB\, dr < \infty \to s > 1 \quad \text{and} \quad 1 < n < 2 \tag{36}$$

where $B(r) = \overline{u(x)u(x+r)}$ is the autocorrelation function.[18]

Going back to Eq. (19), self-preservation of the dissipative end of the spectrum can also be examined under a different selection of the scaling quantities. Following Kolmogorov's scaling, the dissipative range parameters are defined by the energy dissipation rate ϵ and the kinematic viscosity ν. Then

$$U(t) = (\epsilon\nu)^{1/4} \tag{37}$$

$$L(t) = \eta(t) = (\nu^3\epsilon^{-1})^{1/4} \tag{38}$$

When these definitions are substituted in Eq. (19), all terms on the left-hand side cancel out, leaving only

$$0 = \alpha(v_{i-1}^2 \ldots) - \kappa_i^2 v_i \tag{39}$$

and a universal spectrum depending only on the model parameters. With the dissipation rate determined by the large-eddy dynamics, we have

$$\epsilon \sim u^3/\mathcal{L} \sim t^{-3n/2}t^{n/2-1} \sim t^{-(1+n)} \tag{40}$$

so that

$$\eta = \eta_0 t^{(1+n)/4} \tag{41}$$

The classical relation

$$\mathcal{L}/\eta \sim Re_{\mathcal{L}}^{3/4} \tag{42}$$

is thus satisfied.

IV. Decay of Temperature Variance

A variant of the cascade equation [Eq. (6)] applicable to temperature fluctuations was presented by Kerr and Nelkin.[17] It reads

$$\frac{d\theta_i}{dt} = \alpha_\theta k_i[(\mu_{i-1}\theta_{i-1} - 2\theta_{i+1}\mu_i)$$

$$- 2^{1/3}C_\theta(\theta_{i-1}\mu_i - 2\mu_{i+1}\theta_{i+1})] - \frac{\nu}{Pr} k_i^2\theta_i \tag{43}$$

where α_θ and C_θ are model constants, here assumed to be equal to α and C, respectively, and Pr is the Prandtl number. As for their exact counterparts, Eqs. (6) and (43) differ in that, for a given velocity spectrum, the temperature equation is linear in θ. Equation (43) cannot be expected to apply for very large Pr in the range of scales between the velocity and temperature Kolmogorov scales. Indeed, Batchelor[19] observed that the scalar spectral transfer obeys somewhat different dynamics in the viscous-convective range.

The following definitions will be used:

Spectral density:

$$E_{\theta i}(k_i) = \frac{\theta_i^2}{k_i} \tag{44}$$

Dissipation rate:

$$\epsilon_\theta = 2\frac{\nu}{Pr}\int k^2 E_\theta \, dk \tag{45}$$

Kolmogorov-Batchelor constant:

$$E_\theta(k) = \alpha_B \epsilon^{-1/3}\epsilon_\theta k^{-5/3} \tag{46}$$

Root-mean-square fluctuation:

$$\theta' = \left(\int E_\theta \, dk\right)^{1/2} \tag{47}$$

Integral scale:

$$\mathcal{L}_\theta = \frac{\pi}{2\theta'^2}\int \frac{E_\theta}{k} dk \tag{48}$$

Taylor microscale:

$$\lambda_\theta = \left(\frac{6\int E_\theta \, dk}{\int k^2 E_\theta \, dk}\right)^{1/2} \tag{49}$$

The definitions can be applied to the self-similar decay of the temperature spectrum. For scaling quantities, velocity and length scales were discussed in Sec. III and must be used here for consistency. Thus, it is sufficient to define a temperature scale $T(t)$. By analogy with the energy decay, we take

$$\theta_i(k_i,t) = T(t)\cdot\tau_i(\kappa_i) \tag{50}$$

Substitutions of the scaled variables in Eq. (43) and some simple manipulations yield

$$\tau_i\frac{dT}{dt} + T\frac{d\tau_i}{d\kappa_i}\frac{\kappa_i}{L}\frac{dL}{dt} = \alpha\frac{UT}{L}\kappa_i(v_{i-1}\tau_{i-1}\ldots) - \frac{\nu}{Pr}\frac{U}{L^2}\kappa_i^2\tau_i \tag{51}$$

Since power laws of the form

$$\mathcal{L} = \mathcal{L}_0\left(\frac{u}{\mathcal{L}_0}t\right)^{1-n/2}$$

$$u = u_0\left(\frac{u_0}{\mathcal{L}_0}t\right)^{-n/2}$$

$$T = T_0\left(\frac{u_0}{\mathcal{L}_0}t\right)^{-m+1-n/2}$$

are expected, those solutions can be substituted in Eq. (51), yielding

$$-\frac{m}{2} + \left(1-\frac{n}{2}\right)\frac{\kappa_i}{\tau_i}\frac{d\tau_i}{d\kappa_i} = \alpha\frac{\kappa_i}{\tau_i}(v_{i-1}\tau_{i-1}\ldots) - \frac{1}{RePr}\kappa_i^2\tau_i \tag{52}$$

which admit self-similar solutions when the variations of very large Reynolds number are ignored. The small-wave-number asymptotic behavior is found to be

$$\tau_i \sim \kappa_i^{m/(2-n)} \quad \text{as} \quad \kappa_i \to 0 \tag{53}$$

Accordingly, the decay of the temperature variance will preserve the nondissipative part of the spectrum, with a decay exponent that depends on both the velocity and temperature spectra at small wave numbers:

$$E_{\theta i}(k_i,t) \sim k_i^{(2m+n-2)/(2-n)} \sim k_i^q, \qquad k_i \to 0 \tag{54}$$

Finally, m is obtained from Eq. (58):

$$m = 2\frac{(q+1)}{(s+3)} \tag{55}$$

$$r = \frac{m}{n} = \frac{(q+1)}{(s+1)} \tag{56}$$

V. Numerical Results

Although Eq. (27) cannot be integrated analytically, its discrete form

$$\frac{v_{i+1} - v_{i-1}}{\log(\kappa_{i+1}) - \log(\kappa_{i-1})} = \frac{n}{2-n}v_i + \frac{2\alpha}{2-n}\kappa_i$$

$$\times [(v_{i-1}^2 - 2v_{i+1}v_{i+1}v_i) - 2^{1/3}C(v_{i-1}v_i - 2v_{i+1}^2)]$$

$$- \frac{2}{Re(2-n)}\kappa_i^2 v_i \tag{57}$$

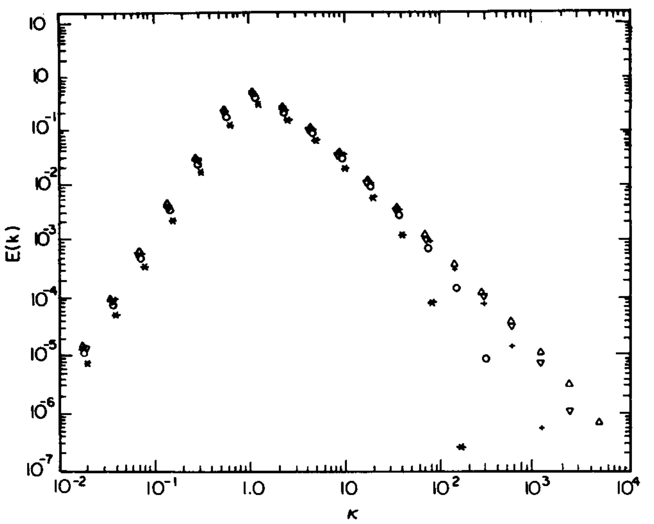

Fig. 1 Dependence of the quasi-self-preserving spectra on the large-eddy Reynolds number for $n = 1.3$, $\alpha = 0.65$, $C = 0.35$; *, $Re = 2.5\times10^2$; ○, $Re = 1.25\times10^3$; +, $Re = 6.25\times10^3$; ▽, $Re = 3.125\times10^4$; Δ, $Re = 1.5625\times10^5$.

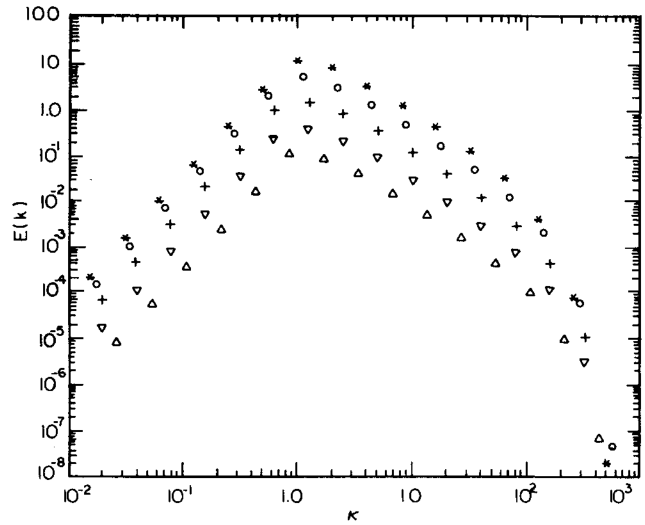

Fig. 3 Dependence of the quasi-self-preserving spectra on the model parameter α for $Re = 1\times10^3$, $n = 1.3$, $C = 0.35$; *, $\alpha = 0.25$; ○, $\alpha = 0.5$; +, $\alpha = 1.0$, ▽, $\alpha = 2.0$; Δ, $\alpha = 4.0$.

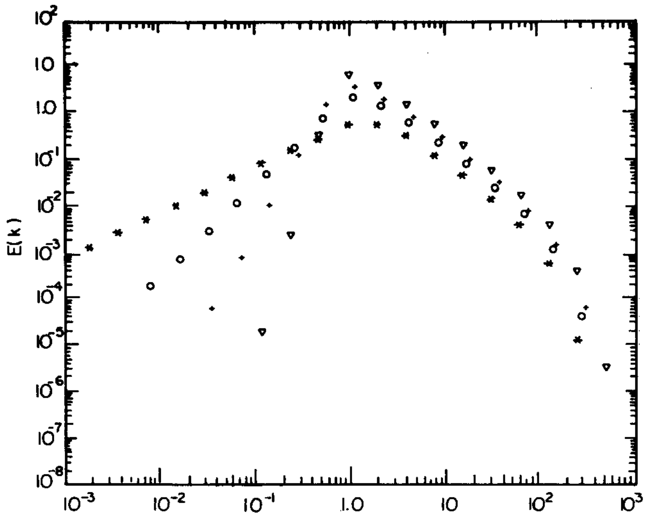

Fig. 2 Dependence of the quasi-self-preserving spectra on the decay exponent for $Re = 1\times10^3$, $\alpha = 0.65$, $C = 0.35$; *, $n = 1$; ○, $n = 1.2$; +, $n = 1.4$; ▽, $n = 1.6$.

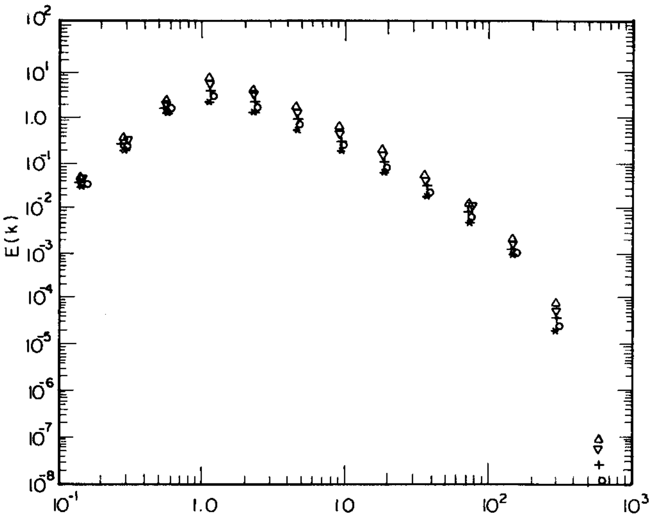

Fig. 4 Dependence of the quasi-self-preserving spectra on the model parameter C for $Re = 1\times10^3$, $n = 1.3$, $\alpha = 0.65$; *, $C = 0.15$; ○, $C = 0.25$; +, $C = 0.35$ ▽, $C = 0.45$; Δ, $C = 0.55$.

lends itself to simple numerical solution. When the viscous dissipation term is constant, the solution to Eq. (57) gives the large-eddy quasi-self-preserving spectrum corresponding to power law decay.

The numerical solution of Eq. (57) was obtained by an iterative scheme analogous to the introduction of an artificial time. Convergence was observed after three to five energy-containing eddy turnover times. This time is small compared to the largest-eddy turnover time. Sample runs lasting as long as 50 time units showed that no further changes to the spectra could REV 1 be observed. Two aspects of the numerical solution can be noted. A simple large-eddy cut-off of the spectrum corresponds to $s \to \infty$. As can be expected from the theory, this asymptotic behavior evolves to dominate the solution and the decay rate associated with it. Thus, the small-κ cut off was treated by power law continuation of the solution. Futhermore, a normalization of the wave-number scale is implied by Eqs. (9) and (11). They yield

$$\sum v_i^2 = \frac{3\pi}{4}\sum \frac{v_i^2}{\kappa_i} \tag{58}$$

Thus, the κ axis was renormalized after each iteration. Superposition of the solutions was obtained for different initial spectra, indicating convergence.

Eq. (57) contains two physical parameters (n and Re) and two model parameters (α and C). The dependence of the solution on the parameters is illustrated on Figs. 1–4. In all cases, the sharp transition between the large eddies and the inertial range was obtained for different initial conditions. Figure 1 shows the spectra for a range of large-eddy Reynolds numbers. The nondissipative part of the spectra superposed almost exactly, the slight shift being attributed to the normalization of the κ axis. The n dependence of the spectra is shown on Fig. 2. The large-eddy power law spectrum is clearly observable in all cases.

The model parameters are explored next. The effect of α and C is observed to correspond to a translation of the spectrum in the plane of the figure (Figs. 3 and 4), without appreciable change in shape. Since nondimensionalization amounts to a similar translation, the dimensional spectrum will be independent of the model parameters.

The quasi-self-similar scalar equation

$$\frac{\tau_{i+1}-\tau_{i-1}}{\log(\kappa_{i+1})-\log(\kappa_{i-1})}=\frac{m}{2-n}\tau_i-\frac{2}{RePr(2-n)}\kappa_i^2\tau_i$$
$$+\frac{2\alpha}{2-n}\kappa_i\,[(v_{i-1}\tau_{i-1}-2\tau_{i+1}v_i)$$
$$-2^{1/3}\,C(v_{i-1}v_i-2v_{i+1}\tau_{i+1})] \qquad (59)$$

is, of course, linear and was solved with the velocity spectrum given by Eq. (57). Since the scalar fluctuations are determined only to within an arbitrary mutliplicative constant, normalization to a unit variance was imposed. A range of values of n and m was explored numerically. The results for $n = 1.2$ and several values of m, presented in Fig. 5, can be considered as typical.

As the most notable feature of the temperature spectra, the shift of the peak toward smaller scales as m increases is now examined in some detail. The observation is in qualitative agreement with the measured spectra of Warhaft and Lumley[2] and with their correlation between the peak of the spectra and the value of m (Fig. 6). However, the range of values of n accessible numerically shows that $\mathcal{L}_\theta/\mathcal{L}$ is not a function of $r = m/n$ and depends on m and n in a more complicated manner. Quantitatively, the numerical results agree remarkably well with the experimental data as far as the slope of the $\mathcal{L}_\theta/\mathcal{L}$ vs m line is concerned. The difference in numerical values could be due to the use of integral scales in the numerical study as opposed to the peak of the spectrum in Warhaft and Lumley's paper. Another possible reason is the model dependence of the integral scales, since these integral scales are mostly affected by the overlap of very-large-eddy and inertial-transfer terms in Eqs. (57) and (59).

For the decay exponents, the ratio of input integral scales is not a control parameter, in confirmation of the study of Sreenivasan et al.[3] and in partial disagreement with the rationale of Warhaft and Lumley.[2] The relation between the length scale ratio and the decay exponents is not of cause and effect, but rather of common initial (quasi-self-preserving) conditions. The length scales of the self-preserving solutions are not determined by their input values, but by the large-eddy spectral exponents.

The data of Sreenivasan et al.[3] can be compared to the analytical and numerical results. The small range of m in their experiments corresponds to a remarkably small scatter of the length scale ratios. They also report a gradual increase of $\mathcal{L}_\theta/\mathcal{L}$ as the turbulence decays. Assuming that departure from self-preservation is not a dominant factor, the only available parameter is the Reynolds number [according to Eq. (25), Re decreases as $t^{-0.2}$ in the experiments]. However, the computational results failed to confirm this hypothesis. For $n = 1.2$ and $m = 1.3$, the gradual reduction of Re from 1×10^3 to 5×10 showed minimal scatter and no definite trend for either $\mathcal{L}_\theta/\mathcal{L}$ or λ_θ/λ.

VI. Discussion

The main result of this study is the relation between the small-wave-number spectral exponents and the decay exponents for decaying isotropic turbulence. For the decay of velocity fluctuations, the result was first obtained by Leith[15]; it is remarkable that the result was omitted from the conclusions of Leith's paper; and it has been largely overlooked in the analysis of experiments. From the few references (e.g., Bell and Nelkin[14]), it appears that the result has been regarded as model dependent. Repeating the argument made in Sec. III, model dependence must be acknowledged for those regions of the spectrum where the spectral transfer terms are dominant (inertial range) or significant (energy-containing range and dissipation range), but not where they are asymptotically negligible (the very-large-eddy range).

Within the limits of the cascade model, the discrepancy between these conclusions and those of Bell and Nelkin[14] must be examined. Those authors studied the evolution of a spectrum starting with only one nonempty mode corresponding to a wave number k_0. In the range of wave numbers smaller than k_0, it stands to reason that the reverse-spectral-transfer parameter C governs the rate at which large eddies will develop. Bell and Nelkin go on to show that the decay rate of self-preserving solutions will also depend on C. Their conclusion conflicts directly with the analysis presented here. Two solutions can be proposed to this problem. First, the existence of solutions other than those associated with power law decay cannot be ruled out. Small departures from power law decay might not be noticeable on Bell and Nelkin's numerical re-

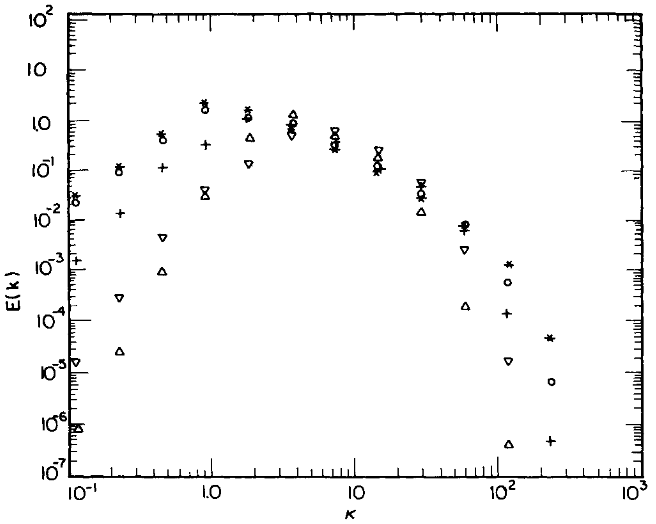

Fig. 5 Quasi-self-preserving scalar spectra for $C_\theta = C = 0.35$, $\alpha_\theta = \alpha = 0.65$, $Re = 1\times10^3$, $Pr = 0.73$, $n = 1.2$; ○, $m = 1.25$; +, $m = 1.75$; ▽, $m = 2.25$; Δ, $m = 2.75$, *, velocity spectrum.

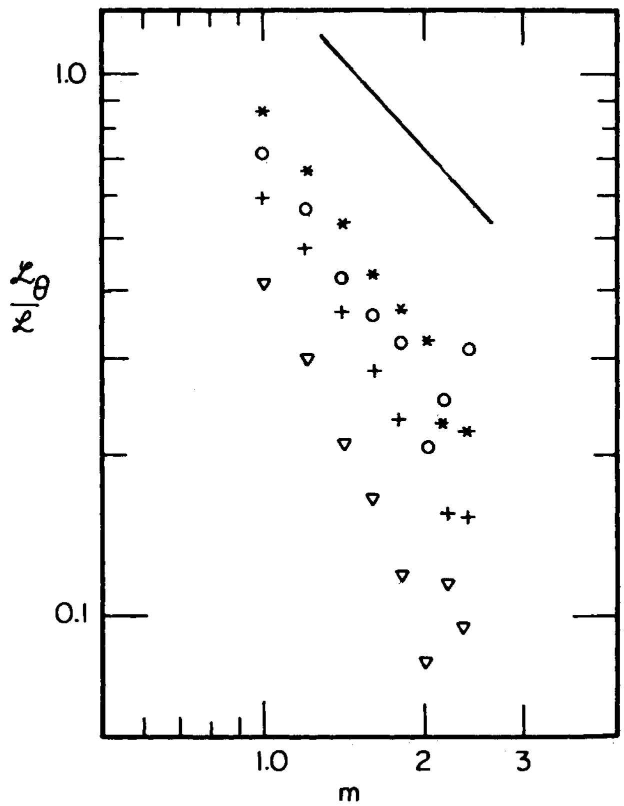

Fig. 6 Dependence of the integral scale ratio on the decay exponent m; *, $n = 1.0$; ○, $n = 1.2$; +, $n = 1.4$; ▽, $n = 1.6$.

sults, which could belong to a different class of solutions altogether. Second, it should be noted that their solution corresponds to an eigenvalue problem for which analytical results could not be obtained. In the absence of details on their numerical procedure, the influence of numerical boundary conditions on the decay rates could explain the results.

Two aspects of the experimental data for grid turbulence are affected by the analysis. A common criterion in the design of experiments has been the comparison of the turbulence integral scale to the tunnel width. If we accept the assumption that grid turbulence is nearly self-preserving in those experiments, the role of the weak, very large eddies has been overlooked. It is doubtful that one order of magnitude separating the integral scale and the tunnel size isolates the decaying turbulence from the tunnel walls. The actual control of the decay exponents presents another difficulty: in a very wide wind tunnel, the large-eddy energy spectrum is determined by the initial conditions, i.e., by the grid configuration. Evidence of this effect was reported by Uberoi and Wallis,[20] but this line of investigation was apparently not pursued further.

In view of the difficulty in making accurate measurements in the large-eddy range, exact simulations provide an alternative to experiments. However, the demands made on the simulation codes are increased by this analysis. The codes will have to provide adequate resolution of the very large eddies, thereby limiting the range of accessible Reynolds numbers. Futhermore, the properties of the codes in the small wave number limit will determine the results and will have to be carefully documented.

The same discussion and conclusions apply to the decay of scalar variance. In the experiments of Warhaft and Lumley,[2] the change in decay exponents is related to the ratio of the integral scales, although the input length scale is not the control parameter. It seems fair to infer that, by changing the heating pattern in the mandoline, those authors were in fact modifying the spectral distribution of temperature fluctuations. The data of Sreenivasan et al.[3] can be interpreted similarly, with differences in apparatus configuration accounting somehow for the different range of spectral exponents.

Quantitative results about integral length scales should be accepted with caution. The numerical scatter in the evaluation of the length scales, especially for large values of n and m, is due to the poor discretization of the wave number scale around the narrow peak in the energy spectrum. The use of interpolation polynomials to improve the accuracy of the numerical integration would be of doubtful value, since the result is also expected to be model dependent.

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